Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of APPLIED MATHEMATICS AND MECHANICS
ELSEVIER
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 57–69

WAVES IN A GRADIENT-ELASTIC MEDIUM WITH SURFACE ENERGY†

V. I. YEROFEYEV and O. A. SHESHENINA
Nizhnii Novgorod
e-mail: wvs@dynamo.nnov.ru

(*Received* 18 *March* 2004)

The propagation of different types of elastic waves in a gradient-elastic medium with surface energy is considered. The dispersion characteristics of longitudinal and shear body waves, Rayleigh surface waves and antiplane shear surface waves, and antiplane shear waves in a layer are analysed in a linear approximation. Antiplane shear surface waves are also investigated taking geometrical non-linearity into account; their modulation instability, which leads to self-modulation and the formation of stationary envelope waves is also considered.

The extension of the classical theory of elasticity by assigning to each point of a continuum the same properties as a rigid body, goes back to the classical publications [1–3]. In the generalized theories of a continuous medium, the interaction between two parts of a body touching an infinitesimal element of the surface, is characterized not only by a force vector, but also by the action of a torque vector. The most general and complete theories of media with a microstructure are given in the papers by Mindlin [4] and Eringen [5].

In solid-state physics, mainly in the study of materials, the concept of structural levels of deformation has become recognised [6, 7]. According to this concept, each point of a rigid body is regarded as a complex system of interacting structures at a lower structural level.

Theories of continua with a microstructure in such hypotheses occupy an intermediate position between the classical theory of elasticity and solid-state physics, which rest on the idea of the existence of structural levels. A point mass in a continuum with a microstructure has a "reasonable" degree of complexity, which enables both the structure of the material (this is inaccessible for the theory of elasticity) and deformation waves (this is inaccessible for the study of materials) to be described.

The theory of gradient elasticity with surface energy was proposed by Vardoulakis and Georgiadis [8] and is based on Mindlin's theory [4]. An isotropic microuniform material is considered in which, first, the relative distortion is equal to zero, second, the mass of the macromaterial of unit macrovolume is zero, and third, a potential energy density function is postulated, which, in addition to the classical components, has additional terms, namely, a Lehr deformation gradient and surface energy.

1. BASIC EQUATIONS

A uniform space with a microstructure is considered. The position of each structural element in this medium is defined by a radius vector in a Cartesian system of coordinates $Ox_1x_2x_3$. We will assume that the micromedium merges with the macromedium, ρ is the density of the macrovolume, and the macromedium occupies a cube with an edge of length $2h$. To solve the problem, we will use the following postulate for the strain energy density function [8]

$$W = \frac{1}{2}\lambda\varepsilon_{qq}\varepsilon_{rr} + \mu\varepsilon_{qr}\varepsilon_{rq} + \mu c(\partial_m\varepsilon_{qr})(\partial_m\varepsilon_{rq}) + \mu b_m\partial_m(\varepsilon_{qr}\varepsilon_{rq}) \qquad (1.1)$$

†*Prikl. Mat. Mekh.* Vol. 69, No. 1, pp. 61–74, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.01.006

where λ and μ are standard Lamé constants, c and b are the moduli of elasticity of the gradient medium $b_m = b\vartheta_m$, $\vartheta_m\vartheta_m = 1$, ∂_m denotes differentiation with respect to the coordinate x_m, $\varepsilon_{qr} = (\partial_r U_q + \partial_q U_r)$ are the components of the strain tensor, U_r are the components of the displacement vector $\mathbf{U}$, and the subscripts q, r and m take values from 1 to 3.

The last term on the right-hand side of Eq. (1.1) relates to the surface energy, since, by Gauss' theorem it can be written in the form

$$\int_\Omega \partial_m(b_m\varepsilon_{qr}\varepsilon_{rq})d\Omega = b\int_S(\varepsilon_{qr}\varepsilon_{rq})(\vartheta_m n_m)dS$$

where n_m are the components of the unit vector of the outward normal to the surface.

Note that the fact that the potential energy density is positive-definite follows from the constraints imposed on the constants of the medium

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad c > 0, \quad -1 < b/c^{1/2} < 1$$

The coefficient c depends on the dimensions of the structural components

$$c = (h/4)^2 \tag{1.2}$$

As was noted above, the relative distortion is equal to zero, and hence the micro-distortion is not an independent function and is equal to

$$\psi_{qr} = \partial_q U_r$$

Since we have eliminated the differences between the microdensity and the macrodensity, the density of the medium ρ is identical with the density of the micromaterial.

The Cauchy stresses and the couple stresses are respectively equal to

$$\tau_{qr} = \partial W/\partial\varepsilon_{qr}, \quad \mu_{qrm} = \partial W/\partial\chi_{qrm} \tag{1.3}$$

where $\chi_{qrm} = \partial_q\psi_{rm} = \partial_q\partial_r U_m$ is the gradient of the microdistortion.

Using (1.1) and (1.3) we can express the stresses and the couple stresses in terms of the components of the strain tensor

$$\tau_{qr} = \lambda\delta_{qr}\varepsilon_{mm} + 2\mu\varepsilon_{qr} + 2\mu b_m(\partial_m\varepsilon_{qr}), \quad \mu_{mqr} = 2\mu[b_m\varepsilon_{qr} + c\varepsilon_{qr,m}] \tag{1.4}$$

We will assume that there are no body forces and no couple body forces. From the variation in the potential energy, taking δU_q as the independent variation, we can obtain the equations of motion and the boundary conditions in stresses in the case of a smooth boundary

$$\begin{gathered}
\partial_q\sigma_{qr} = \rho\ddot{U}_r, \quad \partial_q\mu_{qrm} + \alpha_{rm} = I\ddot{\psi}_{rm} \\
n_r\tau_{rm} - n_q n_r n_m\partial_m\mu_{qrm} - 2n_r(\delta_{ql} - n_q n_l)\partial_l\mu_{qrm} + \\
+ (n_q n_r n_l(\delta_{lq} - n_l n_j)\partial_j - n_q(\delta_{rl} - n_r n_l)\partial_l)\mu_{qrm} + In_r\ddot{\psi}_{rm} = P_m \\
m_q n_r\mu_{qrm} = R_m
\end{gathered} \tag{1.5}$$

A dot denotes a derivative with respect to t, δ_{qr} is the Kronecker delta, $\sigma_{qr} = \tau_{qr} + \alpha_{qr}$ are the components of the overall stress tensor, α_{rm} are the relative stresses, which, in gradient theory, are "constrained" by the surface energy, P_m is the surface force per unit area, R_m is the couple surface force without a torque per unit area and $I = \rho h^2/3$ is the moment of inertia of the microelement.

Using relations (1.4) and (1.5), we obtain the equation of motion in displacements

$$\rho\ddot{\mathbf{U}} = I\Delta\ddot{\mathbf{U}} + (\lambda + 2\mu)\Delta\mathbf{U} + (\lambda + \mu - \mu c\Delta)\mathrm{rotrot}\mathbf{U} - 2\mu c\Delta^2\mathbf{U} \tag{1.6}$$

or, in invariant form

$$\rho\ddot{\mathbf{U}} = I\Delta\ddot{\mathbf{U}} + (\lambda + \mu - \mu c\Delta)\mathrm{graddiv}\mathbf{U} + \mu\Delta\mathbf{U} - \mu c\Delta^2\mathbf{U} \tag{1.7}$$

2. LONGITUDINAL AND SHEAR WAVES

Consider plane longitudinal waves propagating in an unbounded space in the direction of the x_1 axis. The equation describing them can be obtained from Eq. (1.6) by substituting $\mathbf{U} = (u(x_1, t), 0, 0)$. We have

$$(\lambda + 2\mu)u_{,11} - 2\mu c u_{,1111} + I\ddot{u}_{,11} - \rho\ddot{u} = 0 \quad (2.1)$$

The solution of Eq. (2.1) will be sought in the form of a travelling harmonic wave

$$u = Ae^{i(kx_1 - \omega t)} + \text{c.c.} \quad (2.2)$$

where k is the wave number, ω is the frequency and c.c. denotes the complex-conjugate quantity.
Substituting (2.2) into Eq. (2.1), we obtain the dispersion equation

$$(\lambda + 2\mu)k^2 + 2\mu c k^4 - (Ik^2 + \rho)\omega^2 = 0 \quad (2.3)$$

from which we obtain the following explicit relation between the frequency and the wave number

$$\omega = k\sqrt{\frac{c_l^2 + 2cc_\tau^2 k^2}{1 + h^2k^2/3}}, \quad c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_\tau = \sqrt{\frac{\mu}{\rho}} \quad (2.4)$$

where c_1 and c_τ are the velocities with which the longitudinal and shear waves would propagate if there were no microstructure.

We will denote the phase velocity by the letter C, and use a superscript to indicate the type of wave (l for a longitudinal wave, τ for a shear wave, R for a Rayleigh wave and SH for a shear antiplane wave). We will introduce the normalized frequency, the normalized wave number and the normalized phase velocity

$$k_d = k\sqrt{c}, \quad \omega_d = \omega\frac{h}{\sqrt{3}c_\tau}, \quad C_d = \frac{\omega_d}{k_d} \quad (2.5)$$

which henceforth will be used when investigating the different types of waves.
The phase velocity of the longitudinal wave in normalized quantities will look as follows:

$$C_l^d = \sqrt{\frac{(c_l/c_\tau)^2 + 2k_d^2}{3/16 + k_d^2}}$$

For small values of the wave number, when the dimensions of a microelement have no effect on the wave process, there is no dispersion. In this case the phase velocity C^l is identical with the velocity of a longitudinal wave in a classical elastic medium. When $\omega \to \infty$ there is no dispersion, and the asymptotic value of the phase velocity of a longitudinal wave is

$$C^l = \frac{\sqrt{3}}{2\sqrt{2}}c_\tau \approx 0.61c_\tau, \quad C_d^l = \sqrt{2}$$

Note that the Cosserat model of the medium generally does not describe the dispersion of the longitudinal wave while the Le Roux model of the medium, while describing the dispersion, leads to the fact that there is no asymptotic value of the phase velocity as $\omega \to \infty$ [9].

The equation describing the propagation of a plane shear wave is obtained from Eq. (1.6) by substituting $\mathbf{U} = (0, \upsilon(x_1, t), 0)$

$$\mu\upsilon_{,11} - \mu c\upsilon_{,1111} + k\ddot{\upsilon}_{,11} - \rho\ddot{\upsilon} = 0 \quad (2.6)$$

Similarly, as for a longitudinal wave, we obtain the dispersion relation between the frequency and the wave number for dimensional quantities

$$\omega = kc_\tau\sqrt{\frac{1 + ck^2}{1 + h^2k^2/3}} \quad (2.7)$$

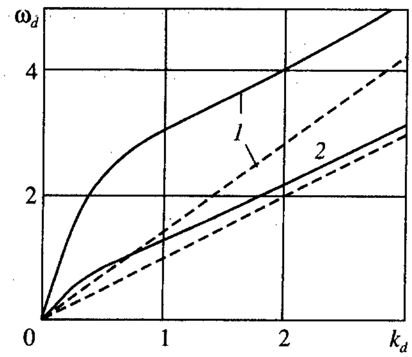

Fig. 1

The phase velocity of a shear wave, normalized using replacement (2.5), will take the following form

$$C_d^\tau = \sqrt{\frac{1+k_d^2}{3/16+k_d^2}}$$

It is found that the dispersion characteristics, constructed form the formulae derived above and from the classical theory of elasticity are identical for small values of k and ω. In this case there is no dispersion, and the phase velocity $C^\tau = c_\tau$. When the frequency increases, the phase velocity decreases and in the limit as $\omega \to \infty$ has the value

$$C^\tau = \frac{\sqrt{3}}{4}c_\tau \approx 0.43c_\tau, \quad C_d^\tau = 1$$

In Fig. 1, for a medium with the parameter $r = \lambda/\mu = 3$ we show graphs of the normalized frequency as a function of the normalized wave number of a longitudinal wave (curve 1) and a shear wave (curve 2) and their asymptotes (the dashed straight lines).

The dispersion equation for a longitudinal wave in normalized quantities only contains only one parameter r, necessary to solve this equation. The form of the dispersion curve for different values of r does not change, but the ratio of the phase velocities of the longitudinal and transverse waves increases as r increases.

Note that, in the equation of motion in displacements (1.6) there are no terms with the parameter b. The velocities of the longitudinal and shear waves are also independent of this parameter, i.e. the additional term in the expression for the potential energy density, responsible for the surface energy, has no effect on the propagation of body waves in the model of the medium investigated.

3. RAYLEIGH SURFACE WAVES

Suppose the half-space occupies the region $x_2 \geq 0$, while the axes of the x_1 and x_3 Cartesian coordinates are directed along the surface.

Consider a plane harmonic wave propagating in the direction of the x_1 axis, whose amplitude decreases exponentially with distance from the free surface $x_2 = 0$. This kind of wave can arise if the perturbation causing it is independent of the variable x_3. The displacement vector will have two non-zero components $U_1(x_1, x_2)$ and $U_2(x_1, x_2)$. In an unbounded space the longitudinal and shear waves propagate independently of one another. The presence of a boundary, as is well known, leads to a coupling between these waves.

Assuming that the $x_2 = 0$ plane is stress-free, we have four conditions

$$\sigma_{21}(x_1, 0) = 0, \quad \sigma_{22}(x_1, 0) = 0, \quad \mu_{221}(x_1, 0) = 0, \quad \mu_{222}(x_1, 0) = 0 \tag{3.1}$$

We will split the displacement vector into potential and solenoidal components

$$\mathbf{U} = \text{grad}\varphi + \text{rot}\phi, \quad \text{div}\phi = 0 \tag{3.2}$$

Since the wave is plane and the motion is independent of x_3, only the component along the x_3 axis: $\phi = (0, 0, \phi_3)$, in the vector potential will be non-zero.

We apply the div operation to Eq. (1.6), and then the rot operation. As a result we obtain the following system of equations

$$\rho\ddot{\varphi} = I\Delta\ddot{\varphi} + (\lambda + 2\mu)\Delta\varphi - 2\mu c\Delta^2\varphi \tag{3.3}$$

$$\rho\ddot{\phi}_3 = I\Delta\ddot{\phi}_3 + \mu\Delta\phi_3 - \mu c\Delta^2\phi_3 \tag{3.4}$$

The solution of each of Eqs (3.3) and (3.4) will consist of two components which decrease with distance from the free surface

$$\begin{aligned}
\varphi &= (Ae^{-\alpha_+ x_2} + Be^{-\alpha_- x_2})e^{i(kx_1 - \omega t)} + \text{c.c.} \\
\phi_3 &= (De^{-p_+ x_2} + Ee^{-p_- x_2})e^{i(kx_1 - \omega t)} + \text{c.c.} \\
\alpha_\pm &= \sqrt{k^2 - z_\pm} \quad p_\pm = \sqrt{k^2 \mp \sigma_\pm^2} \\
z_\pm &= \frac{1}{4\mu c}\left(-\lambda - 2\mu + I\omega^2 \pm \sqrt{(\lambda + 2\mu - I\omega^2)^2 + 8\mu c\rho\omega^2}\right) \\
\sigma_\pm &= \sqrt{\frac{g \mp \chi}{2c}}, \quad g = \sqrt{\chi^2 + 4c\frac{\rho}{\mu}\omega^2}, \quad \chi = 1 - \frac{I}{\mu}\omega^2
\end{aligned} \tag{3.5}$$

where A, B, D and E are amplitude functions.

The boundary conditions, expressed in terms of the components of the displacement vector, will be as follows:

$$\sigma_{21}(x_1, 0) = I\ddot{U}_{1,2} + \mu(U_{2,1} + U_{1,2}) - \mu c\Delta(U_{2,1} + U_{1,2}) = 0$$

$$\begin{aligned}
\sigma_{22}(x_1, 0) &= I\ddot{U}_{2,2} + \lambda(U_{1,1} + U_{2,2}) + 2\mu U_{2,2} - 2\mu c\Delta U_{2,2} = 0 \\
\mu_{221}(x_1, 0) &= \mu b_2(U_{2,1} + U_{1,2}) + \mu c(U_{2,12} + U_{1,22}) = 0
\end{aligned} \tag{3.6}$$

$$\mu_{222}(x_1, 0) = 2\mu b_2 U_{2,2} + 2\mu c U_{2,22} = 0$$

Note that the boundary conditions do not contain the components b_1 and b_3, and hence we can assume that $b_1 = b_3 = 0$ and $b_2 = b \neq 0$.

Substituting expressions (3.5) into (3.2) and using conditions (3.6), we obtain a linear homogeneous system of four equations with unknown amplitude functions A, B, D and E. As is well known, this system has a solution if and only if its determinant is equal to zero. This condition will also be the dispersion relation for a Rayleigh wave.

In order to write the dispersion relation in terms of the normalized frequency and normalized wave number (2.5), the following quantities are necessary, with which it is more convenient to operate henceforth

$$b_d = \frac{b}{\sqrt{c}}, \quad \sigma_{\pm d} = \sigma_\pm\sqrt{c}, \quad z_{\pm d} = z_\pm c \tag{3.7}$$

Then

$$\begin{aligned}
\alpha_{\pm d} &= \sqrt{k_d^2 - z_{\pm d}}, \quad p_{\pm d} = \sqrt{k_d^2 \mp \sigma_{\pm d}^2} \\
z_{\pm d} &= \frac{1}{4}\left(-\frac{\lambda}{\mu} - 2 + \omega_d^2 \pm \sqrt{\left(\frac{\lambda}{\mu} + 2 - \omega_d^2\right)^2 + \frac{3}{2}\omega_d^2}\right) \\
\sigma_{\pm d} &= \sqrt{\frac{g \mp \chi}{2}}, \quad g = \sqrt{\chi^2 + \frac{3}{4}\omega_d^2}, \quad \chi = 1 - \omega_d^2
\end{aligned} \tag{3.8}$$

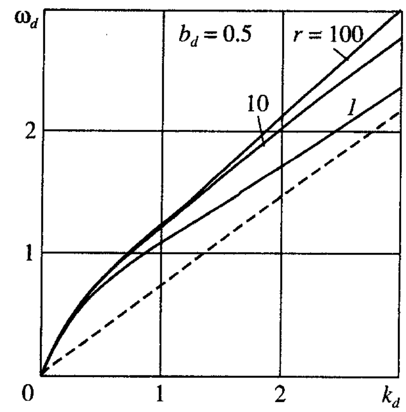

Fig. 2

The dispersion relation in normalized coordinates will take the following form

$$\begin{vmatrix} -2z_{-d}-\frac{3}{16}(C_d^R)^2-\frac{\lambda}{\mu} & -2z_{+d}-\frac{3}{16}(C_d^R)^2-\frac{\lambda}{\mu} & -p_{+d}(1+g) & -p_{-d}(1-g) \\ \alpha_{+d}(\omega_d^2-2-2z_{+d}) & \alpha_{-d}(\omega_d^2-2-2z_{-d}) & k_d^2\left(1+g-\frac{3}{16}(C_d^R)^2\right) & k_d^2\left(1-g-\frac{3}{16}(C_d^R)^2\right) \\ \alpha_{+d}^2(b_d-\alpha_{+d}) & \alpha_{-d}^2(b_d-\alpha_{-d}) & -k_d^2p_{+d}(b_d-p_{+d}) & -k_d^2p_{-d}(b_d-p_{-d}) \\ -2\alpha_{+d}(b_d-\alpha_{+d}) & -2\alpha_{-d}(b_d-\alpha_{-d}) & (k_d^2+p_{+d}^2)(b_d-p_{+d}) & (k_d^2+p_{-d}^2)(b_d-p_{-d}) \end{vmatrix} = 0 \qquad (3.9)$$

As in the case of a longitudinal wave, to solve the dispersion equation it is necessary to specify the ratio $r = \lambda/\mu$. In addition we must assign a value to the parameter b_d.

Dispersion curves were drawn for media with different combinations of values of r and b_d. It turned out that the dispersion curves for media with a fixed value of the parameter r and different values of b_d were so close to one another that their graphs merge. Hence, we can assume that the introduction of the additional term into the expression for the potential energy density, corresponding to the surface energy, has little effect on the nature of the propagation of a Rayleigh surface wave.

In Fig. 2 we show dispersion curves of the normalized frequency as a function of the normalized wave number for $b_d = 0.5$ and different values of r. They all have a common asymptote.

We can conclude from the dispersion curves that the velocity of a Rayleigh wave depends on the frequency, i.e. there is dispersion. If we expand the dispersion equation in a Taylor series in the neighbourhood of $\omega = 0$, it is easy to show that the value of the phase velocity C^R is identical with the value of the phase velocity of a Rayleigh surface wave in the classical theory of elasticity. In a similar way we can find the asymptotic value of the phase velocity as $\omega \to \infty$ for materials with any parameters r and b_d. We present the equation

$$C_d^{R^{14}} - 18C_d^{R^{12}} + 123C_d^{R^{10}} - 406C_d^{R^8} + 757C_d^{R^6} - 748C_d^{R^4} + 356C_d^{R^2} - 64 = 0$$

the root of which is the asymptotic value of the normalized phase velocity. Thus, as $\omega \to \infty$

$$C^R \approx \frac{\sqrt{3}}{4}0.73c_\tau = 0.32c_\tau, \quad C_d^R \approx 0.73$$

4. THE SHEAR ANTIPLANE (SH) SURFACE WAVE

We will consider antiplane shear (i.e. horizontally polarized or SH) motions in a gradient-elastic half-space with surface energy. This type of wave in the model of the medium investigated was considered for the first time by Vardoulakis and Georgiadis [8]. As in the previous problem, suppose the half-space

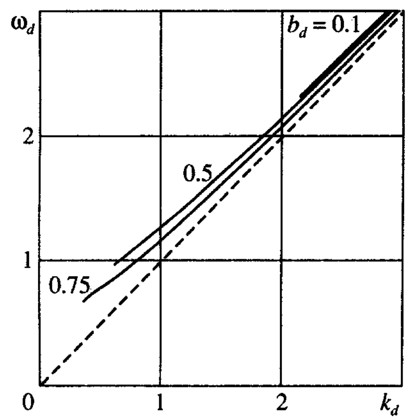

Fig. 3

occupies the region $-\infty < x_1, x_3 < \infty, x_2 \geq 0$. Particles of the medium are displaced in the direction of the x_3 axis. In this case the problem is two-dimensional, and the solution depends only on x_1 and x_2.

We will also assume that $b_1 = b_3 = 0$, $b_2 = b \neq 0$. For the case of SH-motion considered we have $\mathbf{U} = (0, 0, w(x_1, x_2, t))$.

The equation describing the surface SH-wave is identical with the equation for the solenoidal component of the Rayleigh surface wave (3.4) (we must simply replace ϕ_3 by w).

The boundary conditions in displacements have the form

$$\begin{aligned} \sigma_{23}(x_1, 0) &= I\ddot{w}_{,2} + \mu w_{,2} - \mu c w_{,211} - \mu c w_{,222} = 0 \\ \mu_{223}(x_1, 0) &= \mu c w_{,22} + \mu b w_{,2} = 0 \end{aligned} \tag{4.1}$$

The solution of the equation of motion consists of two decreasing components

$$w = [Ae^{-p_+ x_2} + Be^{-p_- x_2}]e^{i(kx_1 - \omega t)} + \text{c.c.} \tag{4.2}$$

where A and B are amplitude functions, and p_+ and p_- satisfy relations (3.5).

The following dispersion relation was obtained in [8]

$$\sigma_{+d}^2\sqrt{k_d^2 - \sigma_{+d}^2} + \sigma_{-d}^2\sqrt{k_d^2 + \sigma_{-d}^2} - b_d(\sigma_{+d}^2 + \sigma_{-d}^2) = 0 \tag{4.3}$$

from which one can express the wave number in terms of the frequency

$$k_d = \sqrt{b_d^2 - \chi + \frac{3}{16}\frac{\omega_d^2}{\chi^2}(\sqrt{b_d^2 - \chi} - b_d)^2} \tag{4.4}$$

Note that relation (4.4) is not satisfied for any ω_d. The start of the range of real frequencies defines the cutoff frequency, which can be found from the condition

$$k_d^2 - \sigma_{+d}^2 = 0 \tag{4.5}$$

Note that the greater the value of the parameter representing the surface energy b_d the wider the frequency range in which these waves can exist.

In Fig. 3 we show graphs of the normalized frequency as a function of the wave number for media with different values of the parameter b_d.

For this problem the additional term in the expression for the potential energy density enables us to prove theoretically that surface SH-waves exist when a uniform medium occupies the half-space. These waves are observed experimentally, for example, in crystal acoustics [10], but they cannot be described within the framework of the classical theory of elasticity. The case considered is a confirmation of the influence of the term in the expression for the potential energy density corresponding to the surface energy.

Surface SH-waves in a half-space of homogeneous material possess dispersion. The dispersion relation in normalized coordinates is independent of the properties of the medium, and hence for all the media the dispersion curves will be similar to the curve shown in Fig. 3. When $\omega \to \infty$ the asymptotic value of the phase velocity is equal to

$$C^{\text{SH}} = \frac{\sqrt{3}}{4}c_\tau \approx 0.43c_\tau, \quad C_d^{\text{SH}} = 1$$

5. THE PROPAGATION OF SH-WAVES IN A LAYER

We will consider shear antiplane wave motions in a layer in the direction of the x_1 axis. The layer is bounded by the planes $x_2 = 0$ and $x_2 = d$. The problem is two-dimensional, and the solution depends only on x_1 and x_2. We will also assume that $b_1 = b_3 = 0$, $b_2 = b \neq 0$. For SH-motions in the layer we have

$$U_1 = U_2 = 0, \quad U_3 = w(x_1, x_2, t) \neq 0$$

The equation of motion will be the same as for surface SH-waves but there will be twice the number of boundary conditions, since there is a second boundary $x_2 = d$ on which conditions similar to (4.1) are imposed.

The solution of the equation of motion will consist of four components

$$w = [A\sin(Px_2) + B\cos(Px_2) + D\,\text{sh}(p_-x_2) + E\,\text{ch}(p_-x_2)]e^{i(kx_1 - \omega t)} + \text{c.c.}$$

$$P = \sqrt{\sigma_+^2 - k^2}$$

where A, B, D and E are amplitude functions.

Introducing the normalized quantities (3.8), similar to the previous case, we arrive at the following dispersion equation

$$\frac{3}{8}\omega_d^2 P_d p_{-d}(1 - \cos(P_d H_d)\text{ch}(p_{-d}H_d)) + \sin(P_d H_d)\text{sh}(p_{-d}H_d)\times$$

$$\times\left(-k_d^2\left(1 - \frac{13}{8}\omega_d^2 + \omega_d^4\right) - 1 + \frac{39}{16}\omega_d^2 - \frac{39}{16}\omega_d^4 + \omega_d^6 + b_d^2 g^2\right) = 0 \qquad (5.1)$$

$$P_d = P\sqrt{c}, \quad H_d = d/\sqrt{c}$$

Dispersion relation (5.1) in normalized coordinates, as in the previous problem, is independent of the properties of the medium. In Fig. 4 we show graphs of the normalized frequency against the normalized wave number for $d = 10h$ and $b_d = 0.5$.

Analysis shows that the zeroth mode of the shear wave in the layer is identical in its dispersion properties to the shear wave in an unbounded gradient-elastic medium given by (2.7).

The asymptotic values of the phase velocities of all modes in the layer are identical with the asymptotic value of the phase velocity of shear and surface SH-waves.

6. NON-LINEAR SURFACE SH-WAVES

We have already considered surface antiplane shear waves in Section 4. Below we investigate the same type of waves, but taking the geometrical non-linearity into account. Unlike the linear problems, we will obtain the equations of motion using the exact expression for the components of the strain tensor

$$\varepsilon_{qr} = (\partial_r U_q + \partial_q U_r + \partial_r U_m \partial_q U_m), \quad r, q, m = 1, 2, 3 \qquad (6.1)$$

Substituting expressions (6.1) into relation (1.1) and taking expressions (1.3) into account, we obtain non-linear relations for the components of the stress tensor and the couple stress tensor

$$\begin{aligned} \tau_{qr} &= \lambda\delta_{qr}\varepsilon_{mm} + 2\mu\varepsilon_{qr} + 2\mu b_m(\partial_m\varepsilon_{qr}) \\ \mu_{mqr} &= 2\mu[b_m\varepsilon_{qr} + c\varepsilon_{qr,m} + (b_m\varepsilon_{qn} + c\varepsilon_{qn,m})U_{r,n}] \end{aligned} \qquad (6.2)$$

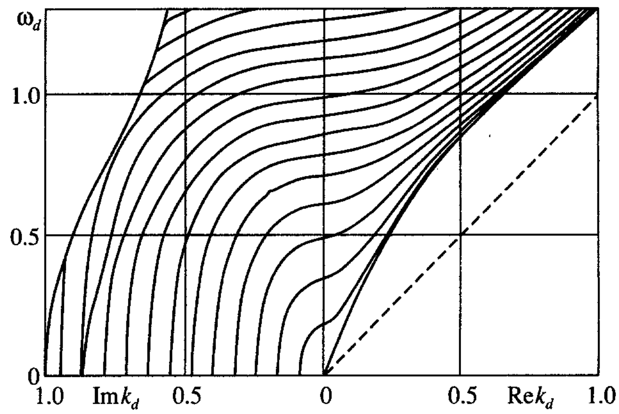

Fig. 4

Further, as in Section 4, we have $\mathbf{U} = (0, 0, w(x_1, x_2, t))$. Substituting expressions (6.2) into Eqs (1.5) we obtain an equation for the antiplane shear components of the displacement

$$\begin{aligned}
&\nabla^2 w - c\nabla^4 w + I\mu^{-1}\nabla^2 \ddot{w} - c_\tau^{-2}\ddot{w} = c\{w_{,1111}(2w_{,1}^2 + w_{,2}^2) + w_{,1122}(3w_{,1}^2 + 3w_{,2}^2) + \\
&+ 2w_{,1}w_{,2}(w_{,1222} + w_{,1112}) + w_{,2222}(w_{,1}^2 + 2w_{,2}^2) + w_{,111}(12w_{,1}w_{,11} + 6w_{,2}w_{,12} + w_{,1}w_{,22}) + \\
&+ w_{,112}(6w_{,2}w_{,11} + 16w_{,1}w_{,12} + 7w_{,2}w_{,22}) + w_{,122}(7w_{,1}w_{,11} + 16w_{,2}w_{,12} + 6w_{,1}w_{,22}) + \\
&+ w_{,222}(6w_{,1}w_{,12} + 12w_{,2}w_{,22} + w_{,2}w_{,11}) + w_{,11}(4w_{,11}^2 + 11w_{,12}^2 + w_{,22}^2) + \\
&+ w_{,22}(4w_{,22}^2 + 11w_{,12}^2 + w_{,11}^2)\} + b\{w_{,112}(3w_{,1}^2 + w_{,2}^2) + 4w_{,122}w_{,1}w_{,2} + \\
&+ w_{,222}(3w_{,2}^2 + w_{,1}^2) + 6w_{,1}w_{,11}w_{,12} + 2w_{,2}w_{,11}w_{,22} + 4w_{,2}w_{,12}^2 + 6w_{,1}w_{,12}w_{,22} + 6w_{,2}w_{,22}^2\}
\end{aligned} \tag{6.3}$$

The conditions on the boundary (4.1) in displacements take the following form

$$\begin{aligned}
&w_{,2} - cw_{,211} - cw_{,222} + I\mu^{-1}\ddot{w}_{,2} = c\{w_{,111}w_{,1}w_{,2} + w_{,112}(w_{,1}^2 + 2w_{,2}^2) + w_{,122}w_{,1}w_{,2} + \\
&+ w_{,222}(w_{,1}^2 + 2w_{,2}^2) + w_{,2}w_{,11}^2 + 3w_{,1}w_{,11}w_{,12} + 5w_{,2}w_{,12}^2 + 3w_{,1}w_{,12}w_{,22} + 4w_{,2}w_{,22}^2\} + \\
&+ b\{w_{,22}w_{,1}^2 + 3w_{,22}w_{,2}^2 + 2w_{,1}w_{,2}w_{,12}\} \\
&-cw_{,22} - bw_{,2} = c\{w_{,1}w_{,2}w_{,12} + w_{,22}w_{,1}^2 + 2w_{,22}w_{,2}^2\} + b\{w_{,2}^3 + w_{,2}w_{,1}^2\}
\end{aligned} \tag{6.4}$$

As already pointed out, in the linear case a solution of the dispersion equation exists if the frequency of the wave exceeds the cutoff frequency, which is found from Eq. (4.5). In Fig. 5 the solution of the above equation is represented by curve 1.

We again consider the linear case, when the solution of the equation of motion consists of two components (4.2). Using conditions (4.1), we conclude that their ratio

$$\frac{Ae^{-p_+x_2}e^{i(kx_1-\omega t)}}{Be^{-p_-x_2}e^{i(kx_1-\omega t)}} = \frac{p_-\sigma_+^2}{p_+\sigma_-^2}e^{-(p_+-p_-)x_2} \tag{6.5}$$

has a minimum value on the boundary $x_2 = 0$, since $p_- > p_+$. The ratio (6.5) decreases as the depth x_2 increases.

Suppose that on the surface the first component is, for example, 20 times greater than the second; points on curve 2 in Fig. 5 satisfy this condition. In the external region of curve 2 the ratio of the two components is always greater than 20. In this case the first component in expression (4.2) predominates over the second, which can henceforth be omitted.

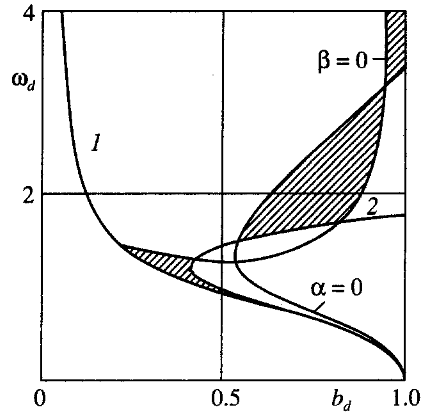

Fig. 5

The solution in the non-linear case will be sought in the form of a single harmonic with a slowly varying complex amplitude A

$$w(x_1, x_2, t) = A(\xi, \tau)e^{-p_+ x_2} e^{i(kx_1 - \omega t)} + \text{c.c.} \tag{6.6}$$

where

$$\xi = \varepsilon\left(x_1 - \upsilon_* t + i\frac{dp_+}{dk}x_2\right), \quad \tau = \varepsilon^2 t$$

$$\frac{dp_+}{dk} = \frac{1}{p_{+d}}\left(k_d - \frac{\sqrt{3}\omega_d \Theta \upsilon_*}{4c_\tau g}\right), \quad \Theta = \frac{16}{3}\sigma_{+d}^2 + 1$$

$$\upsilon_* = \frac{(\sqrt{3}/4)c_\tau \omega_d k_d \chi}{\omega_d^2 \chi + (k_d^2 - b_d^2 + \chi)(1 - b_d \omega_d^2 (b_d^2 - \chi)^{-1/2})}$$

$\upsilon_* = d\omega/dk$ is the group velocity and ε is a small parameter.

Substituting expression (6.6) into Eq. (6.3) and equating coefficients of different powers of the small parameter ε to zero, we obtain for ε^3 a non-linear parabolic Schrödinger equation [11, 12], describing the evolution of the complex amplitude of the quasi-harmonic wave

$$\beta A_{\xi\xi} + iA_\tau - \alpha|A|^2 A = 0 \tag{6.7}$$

where

$$\alpha = \left(\frac{2\sqrt{3}}{hc_\tau}c^2\Theta\omega_d\right)^{-1}(2k_d^6 - 33k_d^4 p_{+d}^2 - 17k_d^2 p_{+d}^4 + 42p_{+d}^6 +$$

$$+ b_d(9k_d^4 p_{+d} - 6k_d^2 p_{+d}^3 - 27p_{+d}^5))e^{-2p_+ x_2}$$

$$\beta = \left(\frac{2\sqrt{3}}{hc_\tau}\Theta\omega_d\right)^{-1}\left\{-g\left(\left(\frac{dp_+}{dk}\right)^2 - 1\right) + \frac{\upsilon_*^2}{c_\tau^2}\Theta\left(-1 - \frac{4}{g}(1-\chi) + \frac{3}{4}\frac{\Theta\omega_d^2}{g^2}\right)\right\}$$

It is well known from the theory of non-linear waves that under certain conditions a quasi-harmonic wave is unstable to splitting into individual wave packets (modulation instability). To answer the question of whether modulation instability of shear surface waves is possible we will use Lighthill's criterion [12], according to which modulation instability is possible in a system in which

$$\alpha\beta < 0 \tag{6.8}$$

It should be noted that the product $\alpha\beta$ is expressed solely in terms of normalized variables, and hence the region of modulation instability will be similar to that derived below for all materials. In Fig. 5 the required region in the plane of the parameters ω_d and b_d is shown hatched.

For different values of the dimensionless parameter b_d there will be one, two or three intervals of the frequencies ω_d for which modulation instability occurs. It should be noted that curves 1 and 2 in Fig. 5 approach one another as b_d increases to unity, but they do not merge. Hence, there is a region of modulation instability between them.

In order to determine the form of the wave packets into which a surface shear wave splits as a result of modulation instability, we will analyse stationary envelope waves. We will introduce a real amplitude a and a real phase φ: $A = ae^{i\varphi}$ instead of a complex amplitude A. Then, Eq. (6.7) can be rewritten in the form of a system of hydrodynamic-type equations

$$\frac{\partial a^2}{\partial \tau} + \frac{\partial a^2 G}{\partial \xi} = 0, \quad \frac{\partial G}{\partial \tau} + G\frac{\partial G}{\partial \xi} - \frac{\tilde{\beta}^2}{2}\frac{\partial}{\partial \xi}\left(\frac{1}{a}\frac{\partial^2 a}{\partial \xi^2}\right) - \frac{\tilde{\alpha}\tilde{\beta}}{2}\frac{\partial a^2}{\partial \xi} = 0$$
$$G = \tilde{\beta}\frac{\partial \varphi}{\partial \xi} \tag{6.9}$$

where $\tilde{\beta} = 2\beta$ is the dispersion parameter and $\tilde{\alpha} = -2\alpha$ is the non-linearity parameter.

We will seek a solution of system (6.9) which depends on the single variable $\eta = \xi - V\tau$, where $V = \text{const}$ is the velocity of the stationary wave. In this case the phase of the wave G can be expressed in terms of its amplitude a

$$G = Q/a^2 + V \tag{6.10}$$

where Q is the constant of integration, while the change in the amplitude is described by the non-linear equation of an harmonic oscillator

$$d^2a/d\eta^2 + m_1 a + m_2 a^3 + m_3 a^{-3} = 0$$
$$m_1 = V^2/\tilde{\beta}^2, \quad m_2 = \tilde{\alpha}/\tilde{\beta}, \quad m_3 = -d^2/\tilde{\beta}^2 \tag{6.11}$$

Note that the coefficient of a is always positive while the coefficient of a^{-3} is always negative. The sign of the coefficient of a^3 can be positive or negative depending on the properties of the material and the frequency band. A positive value of this coefficient will correspond to the regions of modulation instability.

Changing to the new variables

$$\zeta = \sqrt{m_1}\eta, \quad f = \sqrt{m_2/m_1}a \tag{6.12}$$

we will write the first integral of Eq. (6.11) in the form

$$(df/d\zeta)^2 + \Pi(f) = E; \quad \Pi(f) = f^2 + f^4/2 - Df^{-2}, \quad D = m_3 m_2^2/m_1^3$$

where E is the constant of integration.

Analytical solutions of Eq. (6.11) have been obtained and analysed in [13]. The amplitude of the envelope waves is described by the expression

$$f(\zeta) = \pm\sqrt{-\frac{2}{3} + 2A_0\frac{1+s^2}{3s^2} - 2A_0\,\mathrm{sn}^2(k_0\zeta, s)} \tag{6.13}$$
$$A_0 = \frac{R_1 - R_2}{2}, \quad k_0 = \sqrt{\frac{R_1 - R_3}{2}}, \quad s^2 = \frac{R_1 - R_2}{R_1 - R_3}$$

Here A_0 is the amplitude of the stationary envelope wave, k_0 is the analogue of the wave number and s is the modulus of the elliptic function. We have denoted the roots of the polynomial $ER - R^2 - R^3/2 + D$ by R_i $(R_3 \le R_2 \le R_1)$.

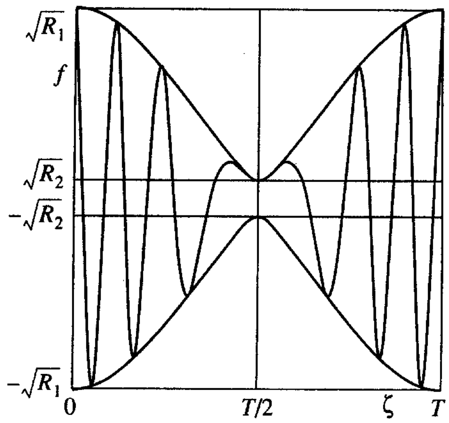

Fig. 6

Solution (6.13) describes periodic motions, the form of which, in general, is non-sinusoidal and is determined by the modulus of s, $s \in (0, 1/2)$. The period of quasi-harmonic oscillations is equal to $T = 2\mathbf{K}(s)/k_0$, where $\mathbf{K}(s)$ is the complete elliptic integral of the first kind.

The periodic sequence of wave packets, into which the shear wave is split as a result of modulation instability, is shown qualitatively (for $s^2 \approx 1/2$) in Fig. 6.

In the special case when $D = 0$, Eq. (6.11) is a Duffing equation, the solution of which in the variables (6.12).

$$f(\zeta) = \sqrt{-1 + \sqrt{1 + 2E}}\operatorname{cn}((1 + 2E)^{1/4}\zeta, s)$$

is close to sinusoidal when $s^2 \to 0$ and has a saw-tooth form when $s^2 \to 1/2$.

Hence, a quasi-harmonic shear wave, modulated periodically, is described by the expression

$$w(x_1, x_2) = \pm\frac{V}{2\sqrt{-\alpha\beta}}Ge^{-p_+x_2}e^{i(kx_1 - \omega t + \varphi)} + \text{c.c.}$$

where

$$\varphi = \frac{V}{2\beta}\eta - \frac{2Q\alpha}{V^2}\int\frac{d\eta}{G^2}$$

and G is the radical on the right-hand side of Eq. (6.13) ($\zeta = V\eta/(2\beta)$).

Stationary envelope waves can also formally exist when there is no modulation instability (this situation was considered in [14]), but the mechanism by which they are formed is still not clear.

7. CONCLUSION

Thus, at zero frequency for longitudinal, shear and SH-waves in a layer and for surface waves, the values of the phase velocities are identical with the corresponding values of the phase velocities, calculated using the classical theory of elasticity. As also in the classical theory of elasticity, the dispersion curves of the zeroth mode of an antiplane (SH) wave in a layer and a body shear wave coincide with one another. The dispersion equations for all the types of waves considered here can be written in terms of the normalized frequency and normalized wave number (2.5). The dispersion relations obtained, in addition, can only include the parameters $r = \lambda/\mu$ and b_d. Because of the additional term in the expression for the potential energy density, with which the parameter b_d is connected, the existence of antiplane (SH) surface waves is proved. This term has only a weak effect on the dispersion relations of the other types of waves. Although the parameter r introduces certain quantitative changes into the dispersion curves, their form remains unchanged.

We will present the asymptotic values of the phase velocities for all the wave modes considered, expressed in terms of the velocity of the shear body wave, known from the classical theory of elasticity (the values corresponding to the classical theory of elasticity are shown in parentheses)

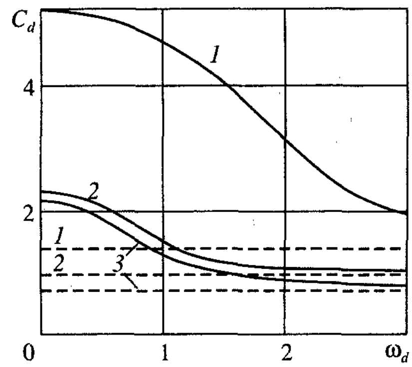

Fig. 7

longitudinal wave	$0.61c_\tau$	(c_l)
shear wave	$0.43c_\tau$	(c_τ)
Rayleigh surface wave	$0.32c_\tau$	(c_R^{cl})
shear antiplane surface wave	$0.43c_\tau$	non-existent
shear antiplane wave in a layer	$0.43c_\tau$	(c_τ)

In Fig. 7 we show curves of the normalized phase velocity as a function of the normalized frequency for a longitudinal wave (curve 1), a shear wave (curve 2) and a Rayleigh wave (curve 3) and their asymptotes for $b_d = 0.5$ and $r = 3$. In the scale of Fig. 7 the curve for the surface SH-wave coincides with curve 2.

If the non-linear terms are taken into account in the equation for the surface antiplane shear motion, at certain frequencies this can lead to modulation instability and the existence of stationary envelope waves.

This research was supported financially by the Russian Foundation for Basic Research (03-02-16924).

REFERENCES

1. VAIGT, W., Theoretische studient über die elasticitatsverhälhnisse der Krystalle. *Abn. Gess. Wess. Gottingen* 1887, **34**.
2. COSSERAT, E. and COSSERAT, F., *Theorie des Corps Deformables*, Hermann, Paris, 1909.
3. LE ROUX. Etude géométrique de la torsion et de la flexion. *Ann. L'École Norm. Sup. Paris* 1911, **28**.
4. MINDLIN, R. D., Micro-structure in linear elasticity. *Arch. Ration. Mech. Analysis* 1964, **16**, 51–78.
5. ERINGEN, A. C., Theory of micropolar elasticity. *Fracture* (Edited by H. Liebowitz), Vol. 2. 1968. 621–729.
6. PANIN, V. Ye., LIKHACHEV, V. A. and GRINYAYEV, Yu. V., *Structural Levels of Deformation of Solids*. Nauka, Novosibirsk, 1985.
7. PANIN, V. Ye. (Ed.), *Physical Mesomechanics in the Computer Construction of Materials*. Izd. MGU, Moscow, 1909.
8. VARDOULAKIS, I. G. and GEORGIADIS, H. G., SH surface waves in a homogeneous gradient-elastic half-space with surface energy. *J. Elasticity*, 1997, **47**, 147–165.
9. YEROFEYEV, V. I., *Wave Processes in Solids with a Microstructure*. Izd. MGU, Moscow, 1999.
10. KRASIL'NIKOV, V. A. and KRYLOV, V. V., *Introduction to Physical Acoustics*. Nauka, Moscow, 1984.
11. RABINOVICH, M. I. and TRUBVETSKOV, D. I., *Introduction to the Theory of Oscillations and Waves*. Nauka, Moscow, 1984.
12. WITHAM, Y. B., *Linear and Nonlinear Waves*. Wiley, New York, 1974.
13. YEROFEYEV, V. I., KAZHAYEV, V. V. and SEMERIKOVA, N. P., *Waves in Rods: Dispersion. Dissipation. Non-linearity*. Fizmatlit, Moscow, 2002.
14. EROFEYEV, V. I. and SEMERIKOVA, N. P., Nonlinear modulated waves in the Timoshenko beam. *Wave Mechanical Systems* (Edited by K. Ragulskis). Technologija, Kaunas, 1996, 12–15.

Translated by R.C.G.